

# Semiclassical Newtonian Field Theories Based On Stochastic Mechanics II

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## Abstract

Continuing the development of the ZSM-Newton/Coulomb approach to semiclassical Newtonian gravity/electrodynamics [1], we formulate a ZSM-Newton/Coulomb version of the large N approximation scheme proposed by Oriols et al. [2]. We show that this new large N scheme makes it possible to self-consistently describe the center-of-mass evolution of a large number of gravitationally/electrostatically interacting, identical, *zbw* particles, without assuming that the particles are weakly coupled, and without entailing the problematic macroscopic semiclassical gravitational/electrostatic cat states characteristic of the mean-field Schrödinger-Newton/Coulomb equations. We also show how to recover N-particle classical Newtonian gravity/electrodynamics for many gravitationally/electrostatically interacting macroscopic particles (composed of many interacting *zbw* particles), as well as classical Vlasov-Poisson mean-field theory for macroscopic particles weakly interacting gravitationally/electrostatically. Finally, we outline an explicit model of environmental decoherence that can be incorporated into Oriols et al.'s scheme as applied to ZSM-Newton/Coulomb.

## 1 Introduction

This paper is a direct continuation of Part I [1]. There, we formulated fundamentally-semiclassical Newtonian gravity/electrodynamics based on stochastic mechanics in the ZSM formulation (ZSM-Newton/Coulomb). Our key results were: (i) ZSM-Newton/Coulomb has a consistent statistical interpretation; (ii) ZSM-Newton/Coulomb recovers the standard quantum description of non-relativistic matter-gravity/charge-field coupling as a special case valid for all practical purposes, even though the gravitational/electrostatic interaction between *zbw* particles is fundamentally classical; and (iii) ZSM-Newton/Coulomb recovers the ‘single-body’ Schrödinger-Newton/Coulomb (SN/SC) and stochastic SN/SC equations as mean-field approximations for systems of gravitationally/electrostatically interacting, identical, *zbw* particles, in the weak-coupling large N limit.

We also discussed some limitations of the mean-field SN/SC and stochastic SN/SC equations: (i) they are based on the assumption that interactions between *zbw* particles are sufficiently weak that the independent particle approximation is plausible; and (ii) the single-body SN/SC and stochastic SN/SC equations admit solutions corresponding to macroscopic semiclassical gravitational/electrostatic cat states, and these cat states predict unphysical gravitational/electrostatic forces on external probe masses. (We also pointed out that the latter difficulty afflicts any formulation of fundamentally-semiclassical Newtonian gravity/electrodynamics based on the many-body SN/SC and stochastic SN/SC equations, as these equations also allow for such cat states.)

The primary objective of the present paper is to develop a new large N scheme for ZSM-Newton/Coulomb, that bypasses the limitations of the mean-field SN/SC and stochastic SN/SC equations.

Our scheme will be based on the one developed recently by Oriols et al. [2], who consider the center-of-mass (CM) motion of a system of N identical, non-relativistic, de Broglie-Bohm (dBB) particles coupled through interaction potentials of the form  $\hat{U}_{int}(\hat{x}_j - \hat{x}_k)$  and to external potentials of the form  $\hat{U}_{ext}(\hat{x}_j)$ . They show

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that, in the limit  $N \rightarrow \infty$ , the CM motion becomes effectively indistinguishable from classical Hamilton-Jacobi mechanics for a single massive particle in an external field.

Essentially, we will import the Oriols et al. scheme into ZSM-Newton/Coulomb. In doing so, we will find that it is possible to: (i) self-consistently describe the CM motion of large numbers of classical-gravitationally/electrostatically interacting, identical, *zbw* particles, without an independent particle approximation; (ii) avoid macroscopic semiclassical gravitational/electrostatic cat states and recover many-particle classical Newtonian gravity/electrodynamics for the CM descriptions of gravitationally/electrostatically interacting macroscopic particles (where the macroscopic particles are composed of many interacting *zbw* particles); and (iii) recover classical Vlasov-Poisson mean-field theory for macroscopic particles that interact gravitationally/electrostatically, in the weak-coupling large particle number limit. We will also be led to suggest an explicit model of environmental decoherence that's consistent with the Oriols et al. scheme, and which could justify a crucial assumption of the scheme.

The outline of the paper is as follows. Section 2 implements the Oriols et al. scheme into ZSM-Newton/Coulomb, and shows how classical Newtonian dynamics for the center-of-mass of a many-particle system is recovered in the large N limit. Section 3 shows how to derive the classical nonlinear Schrödinger equation for the large N center-of-mass motion. Section 4 shows how to recover classical Newtonian gravity/electrodynamics for many gravitationally/electrostatically interacting macroscopic particles. Section 5 shows how to recover classical Vlasov-Poisson mean-field theory. Section 6 sketches an explicit model of environmental decoherence that's consistent with the Oriols et al. scheme applied to ZSM-Newton/Coulomb.

## 2 Large N center-of-mass approximation in ZSM-Newton/Coulomb

### 2.1 General approach

We begin by considering ZSM for N identical *zbw* particles in (for simplicity) 1-dimensional space, with configuration  $X(t) = \{x_1(t), \dots, x_N(t)\}$  and the ensemble-averaged, time-symmetric, joint *zbw* phase

$$\begin{aligned} J(X, t) &:= \int_{\mathbb{R}^N} d^N X \rho(X, t) \int_{t_I}^{t_F} \left\{ \sum_{i=1}^N \frac{1}{2} \left[ 2mc^2 + \frac{1}{2}m(Dx_i(t))^2 + \frac{1}{2}m(D_*x_i(t))^2 \right] - U(X(t), t) \right\} dt + \sum_{i=1}^N \phi_i, \\ &= \int_{\mathbb{R}^N} d^N X \rho(X, t) \int_{t_I}^{t_F} \left\{ \sum_{i=1}^N \left[ mc^2 + \frac{1}{2}mv_i^2(X(t), t) + \frac{1}{2}mu_i^2(X(t), t) \right] - U(X(t), t) \right\} dt + \sum_{i=1}^N \phi_i \end{aligned} \quad (1)$$

where

$$Dx_i(t) = \left[ \frac{\partial}{\partial t} + \sum_{i=1}^N \mathbf{b}_i(X(t), t) \frac{\partial}{\partial x_i} + \sum_{i=1}^N \frac{\hbar}{2m} \frac{\partial^2}{\partial x_i^2} \right] x_i(t), \quad (2)$$

$$D_*x_i(t) = \left[ \frac{\partial}{\partial t} + \sum_{i=1}^N \mathbf{b}_{i*}(X(t), t) \frac{\partial}{\partial x_i} - \sum_{i=1}^N \frac{\hbar}{2m} \frac{\partial^2}{\partial x_i^2} \right] x_i(t). \quad (3)$$

The potential  $U(X(t), t)$  is assumed to take the general form

$$U(X(t), t) := \sum_{j=1}^N U_{ext}(x_j(t)) + \frac{1}{2} \sum_{j,k=1}^{N(j \neq k)} U_{int}(x_j(t) - x_k(t)), \quad (4)$$

and we assume the usual constraints

$$\mathbf{v}_i := \frac{1}{2} [\mathbf{b}_i + \mathbf{b}_{i*}] = \frac{1}{m} \frac{\partial S}{\partial x_i}, \quad (5)$$

$$\mathbf{u}_i := \frac{1}{2} [\mathbf{b}_i - \mathbf{b}_{i*}] = \frac{\hbar}{2m} \frac{1}{\rho} \frac{\partial \rho}{\partial x_i}. \quad (6)$$

As a consequence of (5-6), the time-reversal invariant joint probability density  $\rho(X, t)$  evolves by

$$\frac{\partial \rho}{\partial t} = - \sum_{i=1}^N \frac{\partial}{\partial x_i} \left( \frac{\rho}{m} \frac{\partial S}{\partial x_i} \right), \quad (7)$$

and satisfies the normalization

$$\int_{\mathbb{R}^N} \rho_0(X) d^N X = 1. \quad (8)$$

The stochastic differential equations of motion for  $x_i(t)$  take the form

$$dx_i(t) = \left[ \frac{1}{m} \frac{\partial S}{\partial x_i} + \frac{\hbar}{2m} \frac{1}{\rho} \frac{\partial \rho}{\partial x_i} \right] \Big|_{x_j=x_j(t)} dt + dW_i(t), \quad (9)$$

$$dx_i(t) = \left[ \frac{1}{m} \frac{\partial S}{\partial x_i} - \frac{\hbar}{2m} \frac{1}{\rho} \frac{\partial \rho}{\partial x_i} \right] \Big|_{x_j=x_j(t)} dt + dW_{i*}(t), \quad (10)$$

where the  $dW_i$  ( $dW_{i*}$ ) are 1-dimensional Wiener processes satisfying Gaussianity, independence of  $dx_i(s)$  for  $s \leq t$ , and variance

$$E_t [dW_i^2] = \frac{\hbar}{m} dt. \quad (11)$$

Analogous conditions apply to the backward Wiener processes.

Note that, since we are considering the case of particle motion in a 1-dimensional space, we can disregard the quantization condition for (5) (we will come back to it later, though, when we consider the case of particle motion in a 3-dimensional Euclidean space).

Now, following Oriols et al. [2], we would like to redefine (1) in terms of the CM position  $x_{cm}(t)$  and relative positions  $\mathbf{y}(t) = \{y_2(t), \dots, y_N(t)\}$  such that no cross terms arise from the Laplacians in  $D$  and  $D_*$ . As shown by Oriols et al. [2], the coordinate transformation

$$x_{cm} := \frac{1}{N} \sum_{i=1}^N x_i, \quad (12)$$

$$y_j := x_j - \frac{(\sqrt{N}x_{cm} + x_1)}{\sqrt{N} + 1}, \quad (13)$$

makes it possible to rewrite the N-particle Schrödinger equation, with potential (4), in terms of  $x_{cm}$  and  $\mathbf{y} = \{y_2, \dots, y_N\}$  without cross terms arising from the Laplacian in the Schrödinger Hamiltonian. Thus, applying (12-13) to (1), we obtain <sup>1</sup>

$$\begin{aligned} J(x_{cm}, \mathbf{y}, t) := & \int_{\mathbb{R}^N} \int_{\mathbb{R}^{N-1}} dx_{cm} d^{N-1} \mathbf{y} \rho(x_{cm}, \mathbf{y}, t) \int_{t_I}^{t_F} \left\{ M c^2 + \frac{m}{4} \left[ \left( \tilde{D} x_{cm}(t) \right)^2 + \left( \tilde{D}_{cm*} x_{cm}(t) \right)^2 \right] \right. \\ & + \frac{m}{4} \sum_{j=2}^N \left[ \left( \tilde{D} y_j(t) \right)^2 + \left( \tilde{D}_* y_j(t) \right)^2 \right] - U \Big\} dt + \phi_{cm} + \phi_{rel} = \int_{\mathbb{R}^N} \int_{\mathbb{R}^{N-1}} dx_{cm} d^{N-1} \mathbf{y} \rho(x_{cm}, \mathbf{y}, t) \\ & \times \int_{t_I}^{t_F} \left[ M c^2 + \frac{1}{2} M (v_{cm}^2 + u_{cm}^2) + \frac{1}{2} m \sum_{j=2}^N (v_j^2 + u_j^2) - U \right] dt + \phi_{cm} + \phi_{rel}, \end{aligned} \quad (14)$$

where  $M = Nm$ , the CM velocities are given by

$$\mathbf{v}_{cm} := \frac{1}{2} [\mathbf{b}_{cm} + \mathbf{b}_{cm*}] = \frac{1}{m} \frac{\partial S(x_{cm}, \mathbf{y}(t), t)}{\partial x_{cm}} \Big|_{x_{cm}=x_{cm}(t)}, \quad (15)$$

$$\mathbf{u}_{cm} := \frac{1}{2} [\mathbf{b}_{cm} - \mathbf{b}_{cm*}] = \frac{\hbar}{2m} \frac{1}{\rho(x_{cm}, \mathbf{y}(t), t)} \frac{\partial \rho(x_{cm}, \mathbf{y}(t), t)}{\partial x_{cm}} \Big|_{x_{cm}=x_{cm}(t)}, \quad (16)$$

the relative velocities are given by

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<sup>1</sup>The proof of this goes along the same lines as Appendix A.1 of Oriols et al. [2].

$$\mathbf{v}_j := \frac{1}{2} [\mathbf{b}_j + \mathbf{b}_{j*}] = \frac{1}{m} \frac{\partial S(x_{cm}(t), \mathbf{y}, t)}{\partial y_j} \Big|_{\mathbf{y}=\mathbf{y}(t)}, \quad (17)$$

$$\mathbf{u}_j := \frac{1}{2} [\mathbf{b}_j - \mathbf{b}_{j*}] = \frac{\hbar}{2m} \frac{1}{\rho(x_{cm}(t), \mathbf{y}, t)} \frac{\partial \rho(x_{cm}(t), \mathbf{y}, t)}{\partial y_j} \Big|_{\mathbf{y}=\mathbf{y}(t)}, \quad (18)$$

and the transformed mean forward/backward derivatives take the form

$$\tilde{D}x_{cm}(t) = \left[ \frac{\partial}{\partial t} + \mathbf{b}_{cm} \frac{\partial}{\partial x_{cm}} + \sum_{j=2}^N \mathbf{b}_j \frac{\partial}{\partial y_j} + \frac{\hbar}{2M} \frac{\partial^2}{\partial x_{cm}^2} + \sum_{j=2}^N \frac{\hbar}{2m} \frac{\partial^2}{\partial y_j^2} \right] x_{cm}(t) = \mathbf{b}_{cm}, \quad (19)$$

$$\tilde{D}_*x_{cm}(t) = \left[ \frac{\partial}{\partial t} + \mathbf{b}_{cm*} \frac{\partial}{\partial x_{cm}} + \sum_{j=2}^N \mathbf{b}_{j*} \frac{\partial}{\partial y_j} - \frac{\hbar}{2M} \frac{\partial^2}{\partial x_{cm}^2} - \sum_{j=2}^N \frac{\hbar}{2m} \frac{\partial^2}{\partial y_j^2} \right] x_{cm}(t) = \mathbf{b}_{cm*}, \quad (20)$$

and

$$\tilde{D}y_j(t) = \mathbf{b}_j, \quad (21)$$

$$\tilde{D}_*y_j(t) = \mathbf{b}_{j*}. \quad (22)$$

Accordingly, the continuity equation (7) becomes

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x_{cm}} (\rho v_{cm}) - \sum_{j=2}^N \frac{\partial}{\partial y_j} [\rho v_j], \quad (23)$$

and the forward stochastic differential equations of motion for  $x_{cm}(t)$  and  $y_j(t)$ , respectively, take the form

$$dx_{cm}(t) = \left[ \frac{1}{M} \frac{\partial S(x_{cm}, \mathbf{y}(t), t)}{\partial x_{cm}} + \frac{\hbar}{2M} \frac{1}{\rho(x_{cm}, \mathbf{y}(t), t)} \frac{\partial \rho(x_{cm}, \mathbf{y}(t), t)}{\partial x_{cm}} \right] \Big|_{x_{cm}=x_{cm}(t)} dt + dW_{cm}(t), \quad (24)$$

and

$$dy_j(t) = \left[ \frac{1}{m} \frac{\partial S(x_{cm}(t), \mathbf{y}, t)}{\partial y_j} + \frac{\hbar}{2m} \frac{1}{\rho(x_{cm}(t), \mathbf{y}, t)} \frac{\partial \rho(x_{cm}(t), \mathbf{y}, t)}{\partial y_j} \right] \Big|_{\mathbf{y}=\mathbf{y}(t)} dt + dW_j(t), \quad (25)$$

The  $dW_{cm}$  and  $dW_j$  are 1-dimensional Wiener processes satisfying Gaussianity, independence of  $dx_{cm}(s)$  and  $dy_j(s)$  for  $s \leq t$ , and variances

$$E_t [dW_{cm}^2] = \frac{\hbar}{M} dt, \quad (26)$$

$$E_t [dW_j^2] = \frac{\hbar}{m} dt, \quad (27)$$

respectively. Analogous relations for the backward stochastic differential equations can be written down as well.

We emphasize that (14) is equivalent to (1), the two being related by the coordinate transformations (12-13). Thus, applying

$$J(x_{cm}, \mathbf{y}, t) = \text{extremal}, \quad (28)$$

we obtain

$$\frac{M}{2} [\tilde{D}_* \tilde{D} + \tilde{D} \tilde{D}_*] x_{cm}(t) + \frac{m}{2} \sum_{j=2}^N [\tilde{D}_* \tilde{D} + \tilde{D} \tilde{D}_*] y_j(t) = - \left[ \frac{\partial}{\partial x_{cm}} U(X, t) \Big|_{X=X(t)} + \sum_{j=2}^N \frac{\partial}{\partial y_j} U(X, t) \Big|_{X=X(t)} \right]. \quad (29)$$

By D'Alembert's principle, the variations  $\delta x_{cm}(t)$  and  $\delta \mathbf{y}(t)$  are independent of each other, and the  $\delta y_j(t)$  are independent for all  $j$ . So (29) separates into the pair

$$\frac{M}{2} [\tilde{D}_* \tilde{D} + \tilde{D} \tilde{D}_*] x_{cm}(t) = -\frac{\partial}{\partial x_{cm}} U(X, t)|_{X=X(t)} = -\sum_{i=1}^N \frac{\partial}{\partial x_i} U_{ext}(x_i)|_{x_i=x_i(t)}, \quad (30)$$

$$\frac{m}{2} \sum_{j=2}^N [\tilde{D}_* \tilde{D} + \tilde{D} \tilde{D}_*] y_j(t) = -\sum_{j=2}^N \frac{\partial}{\partial y_j} U(X, t)|_{X=X(t)} = -\frac{1}{2} \sum_{j=2}^N \sum_{k=1}^{N(j \neq k)} \frac{\partial}{\partial x_j} U_{int}(x_j - x_k)|_{X=X(t)}, \quad (31)$$

and (31) separates into

$$\frac{m}{2} [\tilde{D}_* \tilde{D} + \tilde{D} \tilde{D}_*] y_j(t) = -\frac{\partial}{\partial y_j} U(X, t)|_{X=X(t)}, \quad (32)$$

for all  $j$  from 2, ...,  $N$ . The last equality on the right hand side (rhs) of (30) follows from the fact that the symmetry of  $U_{int}$  implies no net force on the CM, and the observation that  $\partial x_i / \partial x_{cm} = 1$  which follows from inverting (12); the last equality on the rhs of (31-32) follows from the fact that the force on the relative degrees of freedom come only from  $U_{int}$ .

Computing the derivatives on the left sides of (30-31), and removing the evaluation at  $X = X(t)$  on both sides, we obtain

$$M \left[ \partial_t \mathbf{v}_{cm} + \mathbf{v}_{cm} \frac{\partial}{\partial x_{cm}} \mathbf{v}_{cm} - \mathbf{u}_{cm} \frac{\partial}{\partial x_{cm}} \mathbf{u}_{cm} - \frac{\hbar}{2M} \frac{\partial^2}{\partial x_{cm}^2} \mathbf{u}_{cm} + \sum_{j=2}^N \left( \mathbf{v}_j \frac{\partial}{\partial y_j} \mathbf{v}_{cm} - \mathbf{u}_j \frac{\partial}{\partial y_j} \mathbf{u}_{cm} - \frac{\hbar}{2m} \frac{\partial^2}{\partial y_j^2} \mathbf{u}_{cm} \right) \right] = -\frac{\partial}{\partial x_{cm}} U, \quad (33)$$

$$m \sum_{j=2}^N \left[ \partial_t \mathbf{v}_j + \sum_{j=2}^N \left( \mathbf{v}_j \frac{\partial}{\partial y_j} \mathbf{v}_j - \mathbf{u}_j \frac{\partial}{\partial y_j} \mathbf{u}_j \right) - \frac{\hbar}{2M} \frac{\partial^2}{\partial x_{cm}^2} \mathbf{u}_j - \frac{\hbar}{2m} \sum_{j=2}^N \frac{\partial^2}{\partial y_j^2} \mathbf{u}_j + \mathbf{v}_{cm} \frac{\partial}{\partial x_{cm}} \mathbf{v}_j - \mathbf{u}_{cm} \frac{\partial}{\partial x_{cm}} \mathbf{u}_j \right] = -\sum_{j=2}^N \frac{\partial}{\partial y_j} U, \quad (34)$$

where  $\mathbf{v}_{cm}$  ( $\mathbf{v}_j$ ) and  $\mathbf{u}_{cm}$  ( $\mathbf{u}_j$ ) are now velocity fields over Gibbsian ensembles of (CM and relative) particles. Thus, by integrating the positional derivatives on both sides of (33) and (34), respectively, and setting the arbitrary integration constants equal to zero (for simplicity), each equation yields the quantum Hamilton-Jacobi equation in CM and relative coordinates:

$$-\partial_t S = U + \frac{1}{2M} \left( \frac{\partial S}{\partial x_{cm}} \right)^2 - \frac{\hbar^2}{2M} \frac{1}{\sqrt{\rho}} \frac{\partial^2}{\partial x_{cm}^2} \sqrt{\rho} + \sum_{j=2}^N \left[ \frac{1}{2m} \left( \frac{\partial S}{\partial y_j} \right)^2 - \frac{\hbar^2}{2m} \frac{1}{\sqrt{\rho}} \frac{\partial^2}{\partial y_j^2} \sqrt{\rho} \right]. \quad (35)$$

Combining with (35) with (23), the Madelung transformation yields the coordinate-transformed Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \left[ -\frac{\hbar^2}{2M} \frac{\partial^2}{\partial x_{cm}^2} - \frac{\hbar^2}{2m} \sum_{j=2}^N \frac{\partial^2}{\partial y_j^2} + U \right] \psi, \quad (36)$$

where  $\psi(x_{cm}, \mathbf{y}, t) = \sqrt{\rho(x_{cm}, \mathbf{y}, t)} e^{iS(x_{cm}, \mathbf{y}, t)/\hbar}$  is single-valued and smooth (because we're restricted to the configuration space of dimension  $\mathbb{R}^N$ ). As shown by Oriols et al. [2], (36) corresponds to the N-particle Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \left[ -\frac{\hbar^2}{2m} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + U \right] \psi, \quad (37)$$

where  $\psi(X, t) = \sqrt{\rho(X, t)} e^{iS(X, t)/\hbar}$ , under the coordinate transformations (12-13).

From the solution of (36), we can rewrite (24-25) as

$$dx_{cm}(t) = \left[ \frac{\hbar}{M} \text{Im} \frac{\partial}{\partial x_{cm}} \ln \psi(x_{cm}, \mathbf{y}(t), t) + \frac{\hbar}{M} \text{Re} \frac{\partial}{\partial x_{cm}} \ln \psi(x_{cm}, \mathbf{y}(t), t) \right] |_{x_{cm}=x_{cm}(t)} dt + dW_{cm}(t), \quad (38)$$

$$dy_j(t) = \left[ \frac{\hbar}{m} \text{Im} \frac{\partial}{\partial y_j} \ln \psi(x_{cm}(t), \mathbf{y}, t) + \frac{\hbar}{m} \text{Re} \frac{\partial}{\partial y_j} \ln \psi(x_{cm}(t), \mathbf{y}, t) \right] \Big|_{\mathbf{y}=\mathbf{y}(t)} dt + dW_j(t). \quad (39)$$

Given an initial wavefunction  $\psi(x_{cm}, \mathbf{y}, 0)$  and initial trajectories  $\{x_{cm}^h(0), \mathbf{y}^h(0)\}$ , where the  $h$  index labels a particular set of possible initial trajectories, the stochastic kinematical evolution of the CM and relative coordinates can be determined completely.

Let's now consider the 2nd-order time-evolution of the mean trajectories of the CM. Defining

$$Q_{cm} := -\frac{\hbar^2}{2M} \frac{1}{\sqrt{\rho}} \frac{\partial^2}{\partial x_{cm}^2} \sqrt{\rho}, \quad (40)$$

$$\sum_{j=2}^N Q_j := -\frac{\hbar^2}{2m} \sum_{j=2}^N \frac{1}{\sqrt{\rho}} \frac{\partial^2}{\partial y_j^2} \sqrt{\rho}, \quad (41)$$

we can rewrite (33) as

$$M \frac{d^2 x_{cm}(t)}{dt^2} = M \left[ \partial_t \mathbf{v}_{cm} + \mathbf{v}_{cm} \frac{\partial}{\partial x_{cm}} \mathbf{v}_{cm} + \sum_{j=2}^N \mathbf{v}_j \frac{\partial}{\partial y_j} \mathbf{v}_{cm} \right] \Big|_{\substack{\mathbf{y}=\mathbf{y}(t) \\ x_{cm}=x_{cm}(t)}} = -\frac{\partial}{\partial x_{cm}} \left[ U + Q_{cm} + \sum_{j=2}^N Q_j \right] \Big|_{\substack{\mathbf{y}=\mathbf{y}(t) \\ x_{cm}=x_{cm}(t)}}, \quad (42)$$

and the  $j$ -th component of (34) as

$$m \frac{d^2 y_j(t)}{dt^2} = m \left[ \partial_t \mathbf{v}_j + \sum_{j=2}^N \mathbf{v}_j \frac{\partial}{\partial y_j} \mathbf{v}_j + \mathbf{v}_{cm} \frac{\partial}{\partial x_{cm}} \mathbf{v}_j \right] \Big|_{\substack{\mathbf{y}=\mathbf{y}(t) \\ x_{cm}=x_{cm}(t)}} = -\frac{\partial}{\partial y_j} \left[ U + Q_{cm} + \sum_{j=2}^N Q_j \right] \Big|_{\substack{\mathbf{y}=\mathbf{y}(t) \\ x_{cm}=x_{cm}(t)}}. \quad (43)$$

We note that equation (42) corresponds to equation (14) of Oriols et al. [2].

Now, consider  $N$  experimental preparations<sup>2</sup> of a system of  $N$  identical particles, described by (36), each with the same initial wavefunction  $\psi(x_{cm}, \mathbf{y}, 0)$ . For each preparation, there will be a different set of “ $h$ -trajectories” [2], and because of the identity of the particles, they will all have the same marginal probability distribution:

$$\bar{\rho}(y_k, 0) := \frac{1}{M} \sum_{h=1}^M \delta(y - y_k^h(0)), \quad (44)$$

where  $M$  is a very large number of preparations. When  $N \rightarrow \infty$ , the distribution of initial particle positions in a single  $h$ -preparation,

$$P(y_k, 0) := \frac{1}{N} \sum_{h=1}^N \delta(y - y_k^h(0)), \quad (45)$$

fills the entire support of (44), thereby giving

$$\bar{\rho}(y_k, 0) \approx P(y_k, 0) \quad (46)$$

for the vast majority of the  $N$  preparations, where ‘vast majority’ refers to the possible set of  $N$  initial trajectories  $X^h(t) = \{x_1^h(t), \dots, x_N^h(t)\}$  selected according to the initial probability density  $\rho(X, 0) = |\psi(X, 0)|^2$ . The fact that the possible set of initial trajectories is selected randomly according to  $|\psi(X, 0)|^2$  ensures that possible sets of initial trajectories which don't satisfy (46) will be extremely rare; and because the  $|\psi(X, 0)|^2$  distribution is preserved in time by the equivariant evolution given by (23), such possible sets of initial trajectories not satisfying (46) will be extremely rare for all times. Thus, Oriols et al. [2] refer to wavefunctions with probability densities satisfying (46) as “wavefunctions full of particles” (WFPs).

As noted by Oriols et al. [2], however, there are  $N$ -particle wavefunctions which don't satisfy (46). For example, a factorizable wavefunction  $\psi(X, 0) = \prod_{i=1}^N \phi(x_i, 0)$  in general won't be a WFP because it won't have the necessary bosonic or fermionic symmetry requirements to justify the independence of the marginal probability distributions for each  $x_i$  (the only exception being a bosonic wavefunction where all the  $\phi_i$  are equal, such

<sup>2</sup>‘Experimental’ could refer to an actual laboratory experiment or a natural physical process outside laboratories.

as in the mean-field approximation). For another example, wavefunctions with strong quantum correlations between the particles (see equation D3 of Oriols et al. [2] for an example involving an unphysical macroscopic superposition state) won't have a single  $\hbar$ -preparation which, in the limit  $N \rightarrow \infty$ , fills the entire support of (44); however, Oriols et al. [2] argue that "most of the wave functions associated to macroscopic objects fulfill the requirements of a wave function full of particles, i.e. they do not include strong quantum correlations between particles" (page 12). While they don't explain why they argue that most wavefunctions associated to macroscopic objects don't include strong quantum correlations between particles, their expectation can be justified from the following observation: in dBB and stochastic mechanics, macroscopic superposition states (in the real world) arise as a result of decoherence from system-environment interactions [3, 4, 5, 6, 7, 8], and such decoherence is always accompanied by "effective collapse" [3, 4, 5, 6, 7, 8]. Effective collapse being the process whereby a dBB/Nelsonian/ZSM particle (or collection of such particles) composing the system dynamically evolves into one of the effective system wavefunction components of a system-environment entangled state, the latter formed during an environmental decoherence process. Thus effective collapse ensures that, for all practical purposes, only *one* of the components of a system-environment entangled state (i.e., a macroscopic quantum superposition state) will be dynamically relevant to the future motion of a single-particle or multi-particle system coupled to a macroscopic environment. In other words, an environmentally decohered macroscopic object (composed of dBB/Nelsonian/ZSM particles), which are virtually all of the macroscopic objects in the real world (according to dBB and stochastic mechanics), can always be expected to have a many-particle effective wavefunction associated to it corresponding to a WFP. In section 6, we will say more about how environmental decoherence and effective collapse of large  $N$  systems might be modeled within the Oriols et al. scheme. In the mean time, we will continue with assuming pure states that satisfy (46) and thus correspond to WFPs.

Focusing now on the CM motion given by (42) and (38), we shall specify the conditions under which its classical limit is obtained. For convenience, we rewrite (42) as

$$M \frac{d^2 \mathbf{x}_{cm}(t)}{dt^2} = F_U + F_{cm} + F_{rel}. \quad (47)$$

Classicality conditions will be obtained from comparing the  $N$ -dependences of the three forces on the rhs of (47).

First we recall that, because of the symmetry of  $U_{int}$ , its net force on the CM is zero, leaving the only non-zero net force coming from  $U_{ext}$ :

$$F_U := -\frac{\partial}{\partial x_{cm}} \sum_{i=1}^N U_{ext}(x_i) = -\sum_{i=1}^N \frac{\partial U_{ext}(x_i)}{\partial x_i}. \quad (48)$$

Furthermore, if spatial variations of  $U_{ext}$  are much larger than the size of the  $N$ -particle system under consideration<sup>3</sup> (which will typically be the case for classical external potentials on macroscopic lengthscales), then (48) can be approximated as (using  $\partial x_i / \partial x_{cm} = 1$ )

$$F_U = F_{ext} \approx -N \frac{\partial U_{ext}(x_{cm})}{\partial x_{cm}}, \quad (49)$$

which is exact for linear and quadratic potentials as pointed out by Oriols et al. [2]. Thus we have that  $F_U \propto N$ .

Second, Oriols et al. [2] note that the conditional probability distribution for the CM position can be found by considering the probability distribution of  $x_{cm}^h(t)$  for a large number of different  $\hbar$ -trajectories given by  $X^h(t) = \{x_1^h(t), \dots, x_N^h(t)\}$ . For a WFP in the limit  $N \rightarrow \infty$ , the second and third moments of the distribution are zero (see Theorems 9-10 of Appendix D of Oriols et al. [2]). Hence, for very large but finite  $N$ , one can expect a normal distribution for the CM position:

$$\rho(x_{cm}^h(t)) \approx \frac{1}{\sqrt{2\pi}\sigma_{cm}} \exp\left(-\frac{[\bar{x} - x_{cm}^h(t)]^2}{2\sigma_{cm}^2}\right), \quad (50)$$

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<sup>3</sup>More precisely, if, for an  $N$ -particle system with number density of width  $d$ , in the presence of an external potential  $U$  with scale of spatial variation given by  $L(U) = \sqrt{|\frac{U'}{U''}|}$ , we have that  $d \ll L(U)$ . This statement is closely related to the classicality condition used in the Ehrenfest theorem and in the quantum-classical limit scheme of Allori et al. [9, 10, 7], i.e., that the de Broglie wavelength  $\lambda$  of a single-particle wavepacket of width  $\sigma$  (where  $\sigma \geq \lambda$ ) satisfies  $\lambda \ll L(U)$ .



where  $\sigma_{cm}$  is estimated (see Theorem 10 in Appendix D of Oriols et al. [2]) to be given by

$$\sigma_{cm}^2 \leq \frac{\sigma^2}{N}, \quad (51)$$

where  $\sigma^2$  is the variance of the marginal distributions, and where  $\bar{x}$  is the mean position of the CM. The relation between these last two variables can be seen as follows. First, we have (from Corollary 1 and Theorem 4 in Appendix B of Oriols et al. [2]) that

$$\bar{x} \equiv \bar{x}_i = \int_{-\infty}^{\infty} dx x \bar{\rho}(x), \quad (52)$$

where  $\bar{\rho}(x)$  is the marginal probability density of the  $i$ -th particle <sup>4</sup> from which it follows (from Corollary 2 in Appendix B of Oriols et al. [2]) that

$$\sigma^2 \equiv \sigma_i^2 = \int_{-\infty}^{\infty} dx (x - \bar{x}) \bar{\rho}(x). \quad (53)$$

Now, calculating  $Q_{cm}$  and  $F_{cm}$  in terms of  $\rho(x_{cm}^h(t))$ , one obtains

$$Q_{cm} \approx \frac{\hbar}{2M\sigma_{cm}^2} \left( 1 - \frac{[\bar{x} - x_{cm}(t)]^2}{\sigma_{cm}^2} \right), \quad (54)$$

$$F_{cm} \approx -\frac{\partial Q_{cm}}{\partial x_{cm}} \Big|_{x_{cm}=x_{cm}(t)}^{\mathbf{y}=\mathbf{y}(t)} \propto \frac{\hbar}{m\sigma^3} \sqrt{N}, \quad (55)$$

where it is used that  $\bar{x} - x_{cm}(t) \approx \sigma/\sqrt{N}$ . Thus  $F_{cm} \propto \sqrt{N}$ .

Third, since we are dealing with identical particles, we have that  $\rho(x_{cm}, y_2, \dots, y_j, \dots) = \rho(x_{cm}, y_j, \dots, y_2, \dots)$ , and so the force  $F_{rel}$  can be rewritten as

$$F_{rel} := \frac{\hbar^2}{2m} \sum_{j=2}^N \left[ \frac{\partial}{\partial x_{cm}} \left( \frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial y_2^2} \right) \right] \Big|_{x_{cm}=x_{cm}(t)}^{\mathbf{y}=\mathbf{y}(t)}. \quad (56)$$

As emphasized by Oriols et al., the exchange symmetry in  $\rho$  means that a single preparation with  $N \rightarrow \infty$  is equivalent to  $h = \{1, \dots, N\}$  different preparations with  $y_2^h(t)$  approximately filling the entire support of  $\rho$  in the  $y_2$  3-space. Accordingly, the quantum equilibrium distribution for the particles implies that the sum in (56) can be approximated by an integral that's weighted by  $\rho$ :

$$F_{rel} \approx N \frac{\hbar^2}{2m} \int_{y_2} \left[ \rho \frac{\partial}{\partial x_{cm}} \left( \frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial y_2^2} \right) \right] \Big|_{x_{cm}=x_{cm}(t)}^{y_3(t), \dots, y_N(t)} dy_2 \rightarrow 0. \quad (57)$$

That (57) vanishes is due to a symmetric distribution of positive and negative summands (for an explicit proof, see Appendix E of Oriols et al. [2]). Thus  $F_{rel} \approx 0$  as  $N \rightarrow \infty$ .

To summarize, then, in the limit that  $N \rightarrow \infty$  for identical particles, we have

$$\begin{aligned} F_U &\propto N, \\ F_{cm} &\propto \sqrt{N}, \\ F_{rel} &\rightarrow 0. \end{aligned} \quad (58)$$

So it is clear that the classical external force  $F_U$  grows much faster (under the stated conditions) than the two quantum forces, as the number of identical particles interacting through  $U_{int}$  becomes very large. This conclusion does not hold, of course, for the relative degrees of freedom, but since the CM motion is the only one that's relevant on macroscopic length scales, this poses no problem. An interesting consequence of (58) is

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<sup>4</sup>This is defined as  $\bar{\rho}(x) \equiv \bar{\rho}_i(x_i) := \int_{-\infty}^{\infty} dx_1 \dots \int_{-\infty}^{\infty} dx_{i-1} \int_{-\infty}^{\infty} dx_{i+1} \dots \int_{-\infty}^{\infty} dx_N \rho(X)$ . Furthermore, for identical particles, the marginal probability density satisfies  $\bar{\rho}_i(x_i) = \bar{\rho}_j(x_j)$  for  $i \neq j$  (see the proof of Theorem 3 in Appendix B of Oriols et al. [2]).



that quantum uncertainty becomes negligible: between any two preparations of an N-particle system,  $X^h(t)$  and  $X^l(t)$ , the CM trajectories and velocities will be very similar, i.e.  $x_{cm}^h(t) \approx x_{cm}^l(t)$  and  $\mathbf{v}_{cm}^h(t) \approx \mathbf{v}_{cm}^l(t)$ .

How large does N have to be for  $F_{cm}$  and  $F_{rel}$  to become negligible relative to  $F_U$ ? This was addressed by Oriols et al. in numerical simulations [2].

In one simulation (Appendix F of Oriols et al. [2]), an initial N-particle wavefunction for identical particles was constructed from pairs of Gaussian wave packets, with random dispersion and opposite random momenta and central positions (in other words, the packets move towards each other and eventually interfere), under the action of an external linear potential. The linear potential spans a lengthscale of  $\sim 10^{-7}m$ , while the packet widths are only  $\sim 10^{-10}m$ , thereby satisfying the condition that the classical external potential varies over lengthscales much greater than the size of the N-particle system. Half of the initial positions of the N particles were selected randomly according to the probability density of the left packet, the other half according to the probability density of the right packet, and then the evolution of the CM was computed under the influence of the three forces in (47). As a comparison, the classical CM was computed from Newton's law with the linear potential (i.e.,  $F_U$  alone), with the same initial CM position and velocity. The resulting trajectories were compared for  $N = 1$  through  $N = 20$  (see Figure 1 of Oriols et al. [2]). For  $N = 1$ , the relative error between the classical and quantum CM motions increases from zero to 45% in 2 picoseconds; for  $N = 20$ , the relative error increase drops to less than 2% in the same duration. In other words, for  $N = 1$ , the classical and quantum CM motions significantly differ from each other in a very short time, as expected, while for  $N = 20$ , the two CM motions become effectively indistinguishable in a very short time. Moreover, even for N distinguishable particles, under the same conditions, Oriols et al. find that the relative error for  $N = 20$  decreases to around 5% in 2 picoseconds. It is remarkable that, under the stated conditions, relatively few particles are needed to reach the "large N" regime.

As a corollary to the above results, we note that, for the case of a WFP, the CM osmotic velocity is given by

$$\mathbf{u}_{cm} = \frac{\hbar}{2M} \frac{1}{\rho} \frac{\partial \rho}{\partial x_{cm}} \Big|_{x_{cm}=x_{cm}(t)} \approx \frac{\hbar}{2m\sigma\sqrt{N}}, \quad (59)$$

while from (47) and (49) the CM current velocity is found to be

$$\mathbf{v}_{cm} = \frac{1}{M} \frac{\partial S}{\partial x_{cm}} \Big|_{x_{cm}=x_{cm}(t)} \approx -\frac{1}{Nm} \int_{t_0}^t \left[ N \frac{\partial U_{ext}}{\partial x_{cm}} - F_{cm} \right] \Big|_{x_{cm}=x_{cm}(t)} dt' + \mathbf{v}_{cm0}, \quad (60)$$

where the contribution from  $F_{rel}$  is neglected because, as we saw from (57), it rapidly approaches zero in the large N limit. Since  $F_{cm}$  is the only N-dependent term in (60) and scales like  $\sqrt{N}$ , we can see that in the large N limit, the dominant contribution to the CM current velocity will come from  $\partial U_{ext}/\partial x_{cm}$ . Accordingly, in the large N limit, the CM current velocity will dominate the kinematics over the CM osmotic velocity (59). Thus, recalling the forward stochastic differential equation

$$dx_{cm}(t) = \left[ \frac{1}{M} \frac{\partial S}{\partial x_{cm}} + \frac{\hbar}{2M} \frac{1}{\rho} \frac{\partial \rho}{\partial x_{cm}} \right] \Big|_{x_{cm}=x_{cm}(t)} dt + dW_{cm}(t), \quad (61)$$

where  $E_t [dW_{cm}^2] = \frac{\hbar}{M} dt$ , we can see that as  $N \rightarrow \infty$ ,  $E_t [dW_{cm}^2] \rightarrow 0$  and (61) reduces to

$$\frac{dx_{cm}(t)}{dt} \approx \frac{1}{M} \frac{\partial S_{cl}}{\partial x_{cm}} \Big|_{x_{cm}=x_{cm}(t)}. \quad (62)$$

The same follows, of course, for the backward stochastic differential equation.

Extending the above approach to the case of 3-space is formally straightforward, and entails the replacements  $\partial/\partial x_{cm} \rightarrow \nabla_{cm}$ ,  $\partial/\partial y_j \rightarrow \nabla_j$ ,  $x_{cm} \rightarrow \mathbf{R}_{cm}$ ,  $\mathbf{y} \rightarrow \mathbf{r}$ , and inclusion of the quantization relation for the phase field  $S$ :

$$\oint_L \nabla_{cm} S(\mathbf{R}_{cm}, \mathbf{r}, t) \cdot d\mathbf{R}_{cm} + \sum_{j=1}^{N-1} \oint_L \nabla_j S(\mathbf{R}_{cm}, \mathbf{r}, t) \cdot d\mathbf{r}_j = nh. \quad (63)$$

This last ensures that the 3N-dimensional generalizations of (23) and (35) are indeed equivalent to the 3N-dimensional generalization of (36), and that  $\psi(\mathbf{R}_{cm}, \mathbf{r}, t)$  is single-valued with (generally) multi-valued phase.

### 3 Classical nonlinear Schrödinger equation for large N center-of-mass motion

#### 3.1 Oriols et al.'s derivation

What form does the time-dependent Schrödinger equation for  $\psi(x_{cm}, \mathbf{y}, t)$  take in the large N limit? Before presenting our answer, let us first review and critique the answer given by Oriols et al. [2].

Introduce the conditional  $S$  and  $\rho$  functions for the CM by the following definitions:

$$S_{cm}(x_{cm}, t) := S(x_{cm}, \mathbf{y}(t), t), \quad \rho_{cm}(x_{cm}, t) := \rho(x_{cm}, \mathbf{y}(t), t), \quad (64)$$

where  $S_{cm}$  satisfies

$$-\partial_t S_{cm} = \frac{1}{2M} \left( \frac{\partial S_{cm}}{\partial x_{cm}} \right)^2 + U(x_{cm}, \mathbf{y}(t), t) + A, \quad (65)$$

with

$$A := Q_{cm} + \sum_{j=2}^N \left[ \frac{1}{2m} \left( \frac{\partial S_{cm}}{\partial y_j} \right)^2 + Q_j - v_j^h(t) \frac{\partial S_{cm}}{\partial y_j} \right], \quad (66)$$

and where  $\rho_{cm}$  satisfies

$$-\partial_t \rho_{cm} = \frac{\partial}{\partial x_{cm}} \left( \frac{1}{M} \frac{\partial S_{cm}}{\partial x_{cm}} \rho_{cm} \right) + B, \quad (67)$$

with

$$B := - \sum_{j=2}^N \left[ \frac{\partial \rho_{cm}}{\partial y_j} v_j^h(t) - \frac{\partial}{\partial y_j} \left( \frac{1}{m} \frac{\partial S_{cm}}{\partial y_j} \rho_{cm} \right) \right]. \quad (68)$$

Using the Madelung transformation, (65) and (67) can then be combined into the ‘conditional Schrödinger equation’ [11]

$$i\hbar \partial_t \psi_{cm} = -\frac{\hbar^2}{2M} \frac{\partial^2 \psi_{cm}}{\partial x_{cm}^2} - \frac{\hbar^2}{2m} \sum_{j=2}^N \frac{\partial^2 \Psi(x_{cm}, \mathbf{y}, t)}{\partial y_j^2} \Big|_{\mathbf{y}=\mathbf{y}^h(t)} + i\hbar \sum_{j=2}^N v_j^h(t) \frac{\partial \Psi(x_{cm}, \mathbf{y}, t)}{\partial y_j} \Big|_{\mathbf{y}=\mathbf{y}^h(t)} + U(x_{cm}, \mathbf{y}(t), t) \psi, \quad (69)$$

where  $\psi_{cm}(x_{cm}, t) = \sqrt{\rho_{cm}(x_{cm}, t)} e^{iS_{cm}(x_{cm}, t)/\hbar}$  is the ‘conditional wavefunction’ in polar form.

Now, from the earlier observation that the large N limit implies

$$\frac{\partial V}{\partial x_{cm}} \Big|_{x_{cm}^h = x_{cm}^h(t)} \gg \frac{\partial (Q_{cm} + \sum_{j=2}^N Q_j)}{\partial x_{cm}} \Big|_{x_{cm} = x_{cm}(t)}, \quad (70)$$

and noting that

$$0 = \left[ \frac{\partial}{\partial x_{cm}} \left( \frac{1}{2m} \left( \frac{\partial S_{cm}}{\partial y_j} \right)^2 - v_j^h(t) \frac{\partial S_{cm}}{\partial y_j} \right) \right] \Big|_{x_{cm}^h = x_{cm}^h(t)}, \quad (71)$$

it follows that  $A \approx 0$  along the CM trajectory. So (65) effectively corresponds to the classical Hamilton-Jacobi equation for the CM, in the large N limit.

Oriols et al. assert that it is reasonable to assume  $B = 0$ , since this turns (67) into the standard continuity equation for the large N CM Gaussian density (50). Thus (65) and (67) can be combined via the Madelung transformation to get the classical nonlinear Schrödinger equation

$$i\hbar \partial_t \psi_{cl} = \left( -\frac{\hbar^2}{2M} \frac{\partial^2}{\partial x_{cm}^2} + U(x_{cm}, t) - Q_{cm} \right) \psi_{cl}, \quad (72)$$

where  $\psi_{cl}(x_{cm}, t) = \sqrt{\rho_{cl}(x_{cm}, t)} e^{iS_{cl}(x_{cm}, t)/\hbar}$  is the ‘classical wavefunction’ for the CM.

The problem with this derivation, in our view, is that no physical justification is given for why it is reasonable to take  $B = 0$ . The fact that such an assumption turns (67) into the standard continuity equation is of course true, but this doesn’t constitute an explanation for *why* it should be true.

### 3.2 Conditional Madelung equations

To demonstrate effective decoupling of the CM and relative coordinates in the large  $N$  limit, it is convenient to focus first on the relative coordinates.

Consider the conditional Madelung variables  $\rho_{rel}(\mathbf{y}, t) := \rho(x_{cm}(t), \mathbf{y}, t)$  and  $S_{rel}(\mathbf{y}, t) := S(x_{cm}(t), \mathbf{y}, t)$ , with evolution equations

$$\partial_t \rho_{rel} = - \sum_{j=2}^N \frac{\partial}{\partial y_j} \left[ \rho_{rel} \frac{\partial S_{rel}}{\partial y_j} \frac{1}{m} \right] - \frac{\partial}{\partial x_{cm}} \left[ \rho \frac{\partial S}{\partial x_{cm}} \frac{1}{M} \right] \Big|_{x_{cm}(t)} + \left( v_{cm}(t) \frac{\partial \rho}{\partial x_{cm}} \right) \Big|_{x_{cm}(t)}, \quad (73)$$

$$\begin{aligned} -\partial_t S_{rel} &= \sum_{j=2}^N \frac{1}{2m} \left( \frac{\partial S_{rel}}{\partial y_j} \right)^2 + \sum_{j=2}^N \left( -\frac{\hbar^2}{2m} \frac{1}{\sqrt{\rho_{rel}}} \frac{\partial^2 \sqrt{\rho_{rel}}}{\partial y_j^2} \right) \\ &\quad + \frac{1}{2M} \left( \frac{\partial S}{\partial x_{cm}} \right) \Big|_{x_{cm}(t)} - \left( \frac{\hbar^2}{2M} \frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial x_{cm}^2} \right) \Big|_{x_{cm}(t)} - v_{cm}(t) \frac{\partial S}{\partial x_{cm}} \Big|_{x_{cm}(t)} + U \Big|_{x_{cm}(t)}, \end{aligned} \quad (74)$$

where again

$$v_{cm}(t) = \frac{dx_{cm}(t)}{dt} = \frac{1}{M} \frac{\partial S}{\partial x_{cm}} \Big|_{x_{cm}(t), \mathbf{y}(t)}, \quad (75)$$

and

$$U \Big|_{x_{cm}(t)} = \left[ \sum_{j=1}^N U_{ext}(x_j) + \frac{1}{2} \sum_{j,k=1; j \neq k}^N U_{int}(x_j - x_k) \right] \Big|_{x_{cm}(t)} = N U_{ext}(x_{cm}(t)) + \frac{1}{2} \sum_{j,k=1; j \neq k}^N U_{int}(x_j - x_k). \quad (76)$$

We will argue that, in the limit  $N \rightarrow \infty$ , the ‘global’  $S$  and  $\rho$  variables effectively decouple in  $\mathbf{y}$  and  $x_{cm}$ , thereby reducing (73-74) to the corresponding effective Madelung equations for the relative coordinates, and likewise for the conditional CM Madelung equations.

#### 3.2.1 Conditional-to-effective continuity equation

Equation (73) can be rewritten as follows:

$$\begin{aligned} \partial_t \rho_{rel} &= - \sum_{j=2}^N \frac{\partial}{\partial y_j} \left[ \rho_{rel} \frac{\partial S_{rel}}{\partial y_j} \frac{1}{m} \right] - \left[ \frac{\partial}{\partial x_{cm}} \left( \rho \frac{\partial S}{\partial x_{cm}} \frac{1}{M} \right) - v_{cm}(t) \frac{\partial \rho}{\partial x_{cm}} \right] \Big|_{x_{cm}(t)} \\ &= - \sum_{j=2}^N \frac{\partial}{\partial y_j} \left[ \rho_{rel} \frac{\partial S_{rel}}{\partial y_j} \frac{1}{m} \right] - \left[ \frac{1}{M} \frac{\partial S}{\partial x_{cm}} \frac{\partial \rho}{\partial x_{cm}} + \frac{1}{M} \rho \frac{\partial^2 S}{\partial x_{cm}^2} - \frac{1}{M} \frac{\partial S}{\partial x_{cm}} \Big|_{\mathbf{y}(t)} \frac{\partial \rho}{\partial x_{cm}} \right] \Big|_{x_{cm}(t)}. \end{aligned} \quad (77)$$

We claim that, in the limit  $N \rightarrow \infty$ , all the terms in the last bracket on the rhs of (77) contribute only as time-dependent correction factors, and therefore can be dropped.

To see this, recall that when  $N \rightarrow \infty$  we have

$$M \frac{d^2 x_{cm}(t)}{dt^2} \approx - \frac{\partial (N U_{ext}(x_{cm}))}{\partial x_{cm}} \Big|_{x_{cm}(t)} = -N \frac{\partial U_{ext}(x_{cm})}{\partial x_{cm}} \Big|_{x_{cm}(t)}, \quad (78)$$

where  $U_{ext}$  spatially varies on macroscopic scales. Integrating (78) gives

$$\frac{dx_{cm}(t)}{dt} = \frac{1}{M} \frac{\partial S(x_{cm}, \mathbf{y}, t)}{\partial x_{cm}} \Big|_{x_{cm}(t), \mathbf{y}(t)} \approx - \frac{1}{m} \int_{t_0}^t \left( \frac{\partial U_{ext}}{\partial x_{cm}} \right) \Big|_{x_{cm}(t')} dt' + v_{cm}(0) =: \frac{1}{M} \frac{\partial S_{cl}(x_{cm}, t)}{\partial x_{cm}} \Big|_{x_{cm}(t)}, \quad (79)$$

and thus

$$\frac{1}{M} \frac{\partial^2 S(x_{cm}, \mathbf{y}, t)}{\partial x_{cm}^2} \Big|_{x_{cm}(t), \mathbf{y}(t)} \approx - \frac{1}{m} \int_{t_0}^t \left( \frac{\partial^2 U_{ext}}{\partial x_{cm}^2} \right) \Big|_{x_{cm}(t')} dt' =: \frac{1}{M} \frac{\partial^2 S_{cl}(x_{cm}, t)}{\partial x_{cm}^2} \Big|_{x_{cm}(t)}, \quad (80)$$

which we see are effectively independent of  $\mathbf{y}$  and only depend on time.

Note, also, that the equation of motion for the relative positions is given by

$$m \frac{d^2 y_j(t)}{dt^2} = - \frac{\partial}{\partial y_j} \left[ \frac{1}{2} \sum_{j,k=1; j \neq k}^N U_{int} + Q_{cm} + \sum_{j=2}^N Q_j \right]. \quad (81)$$

where

$$\frac{dy_j(t)}{dt} = - \frac{1}{m} \int_{t_0}^t \frac{\partial}{\partial y_j} \left[ \frac{1}{2} \sum_{j,k=1; j \neq k}^N U_{int} + Q_{cm} + \sum_{j=2}^N Q_j \right] |_{\mathbf{y}(t'), x_{cm}(t')} dt' + v_j(0) = \frac{1}{m} \frac{\partial S}{\partial y_j} |_{x_{cm}(t), \mathbf{y}(t)}. \quad (82)$$

In the limit  $N \rightarrow \infty$ , we have

$$m \frac{d^2 y_j(t)}{dt^2} \approx - \frac{\partial}{\partial y_j} \left[ \frac{1}{2} \sum_{j,k=1; j \neq k}^N U_{int} + \sum_{j=2}^N Q_j \right] \quad (83)$$

and

$$\frac{dy_j(t)}{dt} = \frac{1}{m} \frac{\partial S(x_{cm}, \mathbf{y}, t)}{\partial y_j} |_{x_{cm}(t), \mathbf{y}(t)} \approx - \frac{1}{m} \int_{t_0}^t \frac{\partial}{\partial y_j} \left[ \frac{1}{2} \sum_{j,k=1; j \neq k}^N U_{int} + \sum_{j=2}^N Q_j \right] |_{\mathbf{y}(t')} dt' + v_j(0) = \frac{1}{m} \frac{\partial S_{rel}(\mathbf{y}, t)}{\partial y_j} |_{\mathbf{y}(t)}, \quad (84)$$

since the large N CM density (corresponding to a WFP) takes the form (50), implying the effective factorization

$$\lim_{N \rightarrow \infty} \rho(x_{cm}, \mathbf{y}, t) \approx \rho_{cl}(x_{cm}, t) \rho_{rel}(\mathbf{y}, t), \quad (85)$$

which leads to  $Q_{cm}$  taking the  $\mathbf{y}$ -independent form (54). Correspondingly, for all  $j = 2, \dots, N$ , equation (85) implies

$$Q_j(x_{cm}, \mathbf{y}, t) \approx - \left( \frac{\hbar^2}{2m} \frac{1}{\sqrt{\rho_{rel}(\mathbf{y}, t)}} \frac{\partial^2 \sqrt{\rho_{rel}(\mathbf{y}, t)}}{\partial y_j^2} \right) = Q_j(\mathbf{y}, t), \quad (86)$$

which is effectively independent of  $x_{cm}$ .

In other words, in the large N limit, the relative coordinates evolve in time (effectively) independently of the CM coordinate.

Accordingly, it follows that (80) only contributes to (77) an uninteresting time-dependent factor of the form

$$\frac{1}{M} \left[ \rho(x_{cm}, \mathbf{y}, t) \frac{\partial^2 S(x_{cm}, \mathbf{y}, t)}{\partial x_{cm}^2} \right] |_{x_{cm}(t)} \approx \frac{1}{M} \left[ \rho_{cl}(x_{cm}, t) \frac{\partial^2 S_{cl}(x_{cm}, t)}{\partial x_{cm}^2} \right] |_{x_{cm}(t)} \rho_{rel}(\mathbf{y}, t), \quad (87)$$

while (79) along with (85) imply the time-dependent factors

$$\frac{1}{M} \left( \frac{\partial S}{\partial x_{cm}} \frac{\partial \rho}{\partial x_{cm}} \right) |_{x_{cm}(t)} \approx \frac{1}{M} \left( \frac{\partial S_{cl}(x_{cm}, t)}{\partial x_{cm}} \frac{\partial \rho_{cl}(x_{cm}, t)}{\partial x_{cm}} \right) |_{x_{cm}(t)} \rho_{rel}(\mathbf{y}, t), \quad (88)$$

and

$$\frac{1}{M} \left( \frac{\partial S}{\partial x_{cm}} |_{\mathbf{y}=\mathbf{y}(t)} \frac{\partial \rho}{\partial x_{cm}} \right) |_{x_{cm}(t)} \approx \frac{1}{M} \left( \frac{\partial S_{cl}(x_{cm}, t)}{\partial x_{cm}} \frac{\partial \rho_{cl}(x_{cm}, t)}{\partial x_{cm}} \right) |_{x_{cm}(t)} \rho_{rel}(\mathbf{y}, t). \quad (89)$$

Hence, terms (87-89) might as well be dropped from (77), leaving

$$\partial_t \rho_{rel} \approx - \sum_{j=2}^N \frac{\partial}{\partial y_j} \left[ \rho_{rel} \frac{\partial S_{rel}}{\partial y_j} \frac{1}{m} \right], \quad (90)$$

which is just the effective continuity equation for  $\rho_{rel}$ .

### 3.2.2 Conditional-to-effective quantum Hamilton-Jacobi equation

Equation (74) can be rewritten as

$$\begin{aligned} -\partial_t S_{rel} = & \sum_{j=2}^N \frac{1}{2m} \left( \frac{\partial S_{rel}}{\partial y_j} \right)^2 + \frac{1}{2M} \left( \frac{\partial S}{\partial x_{cm}} \right)^2 \Big|_{x_{cm}(t)} - \frac{1}{M} \frac{\partial S}{\partial x_{cm}} \Big|_{x_{cm}(t), \mathbf{y}(t)} \cdot \frac{\partial S}{\partial x_{cm}} \Big|_{x_{cm}(t)} \\ & + \sum_{j=2}^N \left( -\frac{\hbar^2}{2m} \frac{1}{\sqrt{\rho_{rel}}} \frac{\partial^2 \sqrt{\rho_{rel}}}{\partial y_j^2} \right) - \left( \frac{\hbar^2}{2M} \frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial x_{cm}^2} \right) \Big|_{x_{cm}(t)} + U \Big|_{x_{cm}(t)}. \end{aligned} \quad (91)$$

From the arguments in sub-subsection 3.2.1, the large  $N$  limit entails that we can neglect the terms involving  $(\partial S / \partial x_{cm})|_{x_{cm}(t)}$ , since they contribute only as time-dependent factors in (91) and drop out of the equations of motion for the relative coordinates.

Similarly, as noted in subsection 2.1, for large  $N$  the center-of-mass quantum kinetic takes the form (54), which means it contributes only an uninteresting time-dependent phase shift to  $S_{rel}$  in (91). And, as we showed in subsection 2.1, that the CM quantum kinetic takes the  $\mathbf{y}$ -independent form (54) means that the CM quantum kinetic drops out of the equations of motion for the relative positions, i.e., equations (81-82), and might as well also be dropped from (91).

Likewise, in  $U|_{x_{cm}(t)}$ , the external potential component  $\sum_{j=1}^N U_{ext}(x_j) = N U_{ext}(x_{cm})$  will also contribute to  $S_{rel}$  only a time-dependent phase shift, and thus can be dropped as well.

We are thereby left with the effective quantum Hamilton-Jacobi equation

$$-\partial_t S_{rel} \approx \sum_{j=2}^N \frac{1}{2m} \left( \frac{\partial S_{rel}}{\partial y_j} \right)^2 + \sum_{j=2}^N \left( -\frac{\hbar^2}{2m} \frac{1}{\sqrt{\rho_{rel}}} \frac{\partial^2 \sqrt{\rho_{rel}}}{\partial y_j^2} \right) + \frac{1}{2} \sum_{j,k=1; j \neq k}^N U_{int}. \quad (92)$$

Accordingly, we conclude that the ‘global’  $S$  function effectively decomposes as

$$\lim_{N \rightarrow \infty} S(x_{cm}, \mathbf{y}, t) \approx S_{cl}(x_{cm}, t) + S_{rel}(\mathbf{y}, t), \quad (93)$$

where  $S_{cl}(x_{cm}, t)$  evolves autonomously by its effective classical Hamilton-Jacobi equation (equation (65) with  $A \approx 0$ ), and likewise for  $\rho_{cl}(x_{cm}, t)$  (equation (67) with  $B \approx 0$ ). The Madelung transformation involving  $S_{cl}(x_{cm}, t)$  and  $\rho_{cl}(x_{cm}, t)$  then yields the classical nonlinear Schrödinger equation (72).

### 3.3 Comments on the classical nonlinear Schrödinger equation

As is well known [12, 13, 14, 15, 16, 17, 2, 18, 11], (72) can also be formally derived (with  $\hbar$  as a free parameter) from classical statistical mechanics of a single particle in an external scalar potential, in the Hamilton-Jacobi representation. What’s different here is that (72) is an approximate description of the Schrödinger evolution for the CM of an  $N$ -particle system, with potential  $U$  (where the external component spatially varies on scales larger than the size of the  $N$ -particle system), in the limit that  $N \rightarrow \infty$ .

In order to verify the robustness of (72) as an approximation to classical dynamics, Oriols et al. [2] numerically simulated a Gaussian wavepacket, defined by taking the square root of (50) and multiplying by  $\exp(ik_0 x_{cm})$ , evolving by (72) for two cases: a packet in free fall in external potential  $U = 2x_{cm}$ , and a packet oscillating in a harmonic oscillator potential  $U = x_{cm}^2/2$ . In both cases (Figures 2 and 3 of [2]), their simulations confirm that the packets do not disperse over time, and the CM trajectories (for different initial positions) closely mimic the CM trajectories one expects from classical mechanics.

Extending our derivation of (72) to the 3-dimensional case is formally straightforward, and requires inclusion of the quantization condition on the 3-dimensional generalization of the CM conditional phase field as follows:

$$\oint_L \nabla_{cm} S(\mathbf{R}_{cm}, \mathbf{r}(t), t) \cdot d\mathbf{R}_{cm} = \oint_L \nabla_{cm} S_{cm} \cdot d\mathbf{R}_{cm} = nh. \quad (94)$$

This assures that the 3-dimensional version of  $\psi_{cm}$  is single-valued with (generally) multi-valued phase.

A notable advantage of (72) as a ‘large  $N$ ’ approximation is that, in contrast to the mean-field SN and stochastic SN equations, (72) does not admit macroscopic superpositions of CM position states, and so does not predict macroscopic semiclassical gravitational/electrostatic cat states in the case that  $U_{int}$  corresponds to an  $N$ -body Newtonian gravitational/Coulomb potential (e.g., such as in a neutron star or the sun). Basically,

this is because the nonlinearity of (72) means that any pair of solutions,  $\psi_1^{cl}$  and  $\psi_2^{cl}$ , cannot be superposed to form a new solution. Thus, only one positional wavepacket is associated to the evolution of the CM at any time.

## 4 Recovering classical Newtonian gravity for many macro particles

Suppose now that we have  $K$  many-particle systems, with CM masses  $\{M^i, \dots, M^K\}$ , where the  $i$ -th CM ‘particle’ is described by a pair of CM Madelung variables  $\{\rho_{cl}^i, S_{cl}^i\}$  evolving by their own effective Madelung equations. Suppose, further, that these CM particles classically interact via macroscopically long-range classical gravitational (or electrostatic) potentials (i.e., potentials spatially varying on scales much larger than the sizes of the  $N$ -particle systems composing the CM particles). Then the  $K$ -body effective Madelung equations for these gravitationally interacting CM particles, are given by

$$-\partial_t \rho_{cl}^K \approx \sum_{i=1}^K \frac{\partial}{\partial x_{cm}^i} \left( \frac{1}{M^i} \frac{\partial S_{cl}^K}{\partial x_{cm}^i} \rho_{cl}^K \right), \quad (95)$$

$$-\partial_t S_{cl}^K \approx \sum_{i=1}^K \frac{1}{2M^i} \left( \frac{\partial S_{cl}^K}{\partial x_{cm}^i} \right)^2 + \sum_{i=1}^K U^i, \quad (96)$$

where the solution of (95) is a product state of narrow Gaussians

$$\rho_{cl}^K \approx \prod_{i=1}^K \frac{1}{\sqrt{2\pi}\sigma_{cm}^i} \exp \left( -\frac{[\bar{x}^i - x_{cm}^i]^2}{2[\sigma_{cm}^i]^2} \right) =: \prod_{i=1}^K \rho_{cl}^i, \quad (97)$$

the solution of (96) takes the form

$$S_{cl}^K \approx \left[ \sum_{i=1}^K \int p_{cm}^i dx_{cm}^i - \int \left( \sum_{i=1}^K \frac{1}{2M^i} (p_{cm}^i)^2 + \sum_{i=1}^K U^i \right) dt \right] - \sum_{i=1}^K \hbar \phi_{cm}^i =: \sum_{i=1}^K S_{cl}^i, \quad (98)$$

and the potential

$$U^i := \sum_{n=1}^{N^i} U_{ext}^i(x_n^i) + \frac{1}{2} \sum_{j,k=1}^{N^i(j \neq k)} U_{int}^i(x_j - x_k), \quad (99)$$

where

$$\sum_{n=1}^{N^i} U_{ext}^i(x_n^i) \approx N^i U_{ext}^i(x_{cm}^i) := -\frac{M^i}{2} \sum_{l=1}^{K(l \neq i)} \frac{M^l}{|x_{cm}^i - x_{cm}^l|}, \quad (100)$$

using  $(\partial x_n^i / \partial x_{cm}^i) = 1$ .

Notice that, despite the CM ‘particles’ gravitationally interacting via (100), the large- $N$  CM densities form a product state (97). This follows from our assumption that the gravitational potentials sourced by the CM ‘particles’ are (macroscopically) long-range, and therefore vary on distance scales much larger than the sizes of the  $N$ -particle systems composing the CM particles (i.e.,  $\sigma_{cm}^i \ll \sqrt{|U_{ext}^i/U_{ext}^i|} [10, 9, 7]$ ). Recall from subsection 2.1 that this was a necessary condition for showing that the large- $N$  density, corresponding to a WFP, takes the (approximately) Gaussian form of the factors in (97). Moreover, although the *z**bw* phases of the CM ‘particles’ are not physically independent, due to the non-separable potential (100) which physically influences each CM particle via the (approximately) classical equations of motion

$$M^i \frac{dx_{cm}^i(t)}{dt} \approx \frac{\partial}{\partial x_{cm}^i} S_{cl}^K |_{x_{cm}^i = x_{cm}^i(t)}, \quad (101)$$

$$M^i \frac{d^2 x_{cm}^i(t)}{dt^2} \approx -N^i \frac{\partial}{\partial x_{cm}^i} U_{ext}^i(x_{cm}^i, t) |_{x_{cm}^i = x_{cm}^i(t)} = M^i \frac{\partial}{\partial x_{cm}^i} \sum_{l=1}^{K(l \neq i)} \frac{M^l}{2|x_{cm}^i - x_{cm}^l|} \Big|_{x_{cm}^i = x_{cm}^i(t)}^{x_{cm}^l = x_{cm}^l(t)}, \quad (102)$$

it is still meaningful to speak of the *zbw* phase of an individual CM ‘particle’ in the lab frame; the  $i$ -th CM ‘particle’, in the lab frame, has an associated *zbw* phase  $S_{cl}^i$  that depends on the sum of all the potentials sourced by the  $K-1$  other CM ‘particles’, at the space-time location of the  $i$ -th CM ‘particle’. Indeed, the net potential ‘seen’ by an individual CM particle, from the  $K-1$  CM particles, looks like a slowly varying external potential as a consequence of  $\sigma_{cm}^i \ll \sqrt{|U_{ext}^i/U_{ext}^{'''}}|$ . Thus  $S_{cl}^i$  varies slowly as a function of  $U_{ext}^i$  for all  $i = 1, \dots, K$ , much like the phase of a light wave moving through a medium of slowly (spatially) varying refractive index.

If we employ the Madelung transformation, (95-96) can be combined into the K-body version of (72):

$$i\hbar\partial_t\psi_{cl}^K = \sum_{i=1}^K \left( -\frac{\hbar^2}{2M^i} \frac{\partial^2}{\partial x_{cm}^2} + U^i - Q_{cm}^i \right) \psi_{cl}^K, \quad (103)$$

where

$$\psi_{cl}^K \approx \prod_{i=1}^K \sqrt{\rho_{cl}^i} e^{iS_{cl}^i/\hbar} =: \prod_{i=1}^K \psi_{cl}^i, \quad (104)$$

$$Q_{cm}^i := -\frac{\hbar^2}{2M^i} \frac{1}{\sqrt{\rho_{cl}^K}} \frac{\partial^2}{\partial x_{cm}^2} \sqrt{\rho_{cl}^K} \approx \frac{\hbar}{2M^i (\sigma_{cm}^i)^2} \left( 1 - \frac{[\bar{x}^i - x_{cm}^{hi}]^2}{[\sigma_{cm}^i]^2} \right). \quad (105)$$

So the Madelung variables for each large- $N$  CM particle define narrow (in position space) classical wavepackets  $\psi_{cl}^i$  satisfying the nonlinear Schrödinger equation (103).

An important property of the K-body system of large- $N$  CM ‘particles’ is that the CM particle trajectories can cross in configuration space. To see this, let us recall what the exact dBB/Nelsonian dynamics predict for a CM ‘particle’ associated to a pure state  $\Psi$ , when  $\Psi$  is a superposition of two (not necessarily narrow) Gaussian wavepackets in position space moving with fixed speeds in opposite directions towards each other. When the packets overlap in configuration space, the exact description says that an ensemble of identical CM particle trajectories, corresponding to each packet, will not cross but rather will abruptly (but not discontinuously) change directions and exit the overlapping region with the packets they did not initially occupy [14, 19, 7]. The physical reason for this non-classical behavior is that the pure state defines a single-valued momentum field in configuration space through  $p = \hbar M \nabla \ln \Psi$ , which means that there will be a unique momentum for each point in the overlap region. Equivalently, the quantum forces from the quantum kinetic associated to  $\Psi$  in the overlap region push the trajectories away from each other and causes them to abruptly change directions. In the case of the K-body system, a superposition of two wavepackets can’t be applied since the packets associated to the large- $N$  CM ‘particles’ evolve by coupled nonlinear Schrödinger equations (103), and any superposition of two packets doesn’t form a new solution of (103). Nevertheless, we can consider two, identical, large- $N$  CM particles, associated to two narrow Gaussian wavepackets moving in opposite directions towards each other and ask if their trajectories will cross (assume the two particles don’t classically interact or only negligibly so). Yes, because (i) the narrowness of the two packets (recall that  $\sigma_{cm} \equiv \frac{\sigma^2}{N}$ , and we have  $N \rightarrow \infty$ , implying that the amplitudes of the packets are effectively Dirac delta functions) ensures that they are effectively disjoint (hence don’t interfere) in position space, and (ii) the quantum force is absent from the large  $N$  equations of motion (101-102). So the large- $N$  CM ‘particles’ indeed move like classical mechanical particles, since classical mechanics predicts that particle trajectories can cross in configuration space (but not in phase space). Similar observations have been made by Benseny et al. in [20] and Dürr et al. in [7].

## 5 Recovering classical Vlasov-Poisson mean-field theory

We can now connect the K-body system of gravitationally interacting, large- $N$  CM ‘particles’ to the classical Vlasov-Poisson mean-field theory.

Assuming the special case of identical CM ‘particles’, multiplying the first term on the rhs in (99) by  $1/K$  (the weak-coupling scaling CITE), and subtracting out the second term on the rhs in (99) (since it will only yield a global phase factor), the K-body effective CM Madelung equations become

$$-\partial_t \rho_{cl}^K \approx \sum_{i=1}^K \frac{\partial}{\partial x_{cm}^i} \left( \frac{1}{M} \frac{\partial S_{cl}^K}{\partial x_{cm}^i} \rho_{cl}^K \right), \quad (106)$$



$$H_{cl}^K := -\partial_t S_{cl}^K \approx \sum_{i=1}^K \frac{1}{2M} \left( \frac{\partial S_{cl}^K}{\partial x_{cm}^i} \right)^2 - \frac{M^2}{K} \sum_{i=1}^K \sum_{l=1}^{K(l \neq i)} \frac{1}{2|x_{cm}^i - x_{cm}^l|}, \quad (107)$$

with solutions given by (97-98) for  $M^i = M$ . The classical equations of motion are just

$$p_{cm}^i(t) := M \frac{dx_{cm}^i(t)}{dt} \approx \frac{\partial}{\partial x_{cm}^i} S_{cl}^K |_{x_{cm}^i = x_{cm}^i(t)}, \quad (108)$$

$$\frac{dp_{cm}^i(t)}{dt} = M \frac{d^2 x_{cm}^i(t)}{dt^2} \approx \frac{M^2}{K} \frac{\partial}{\partial x_{cm}^i} \sum_{l=1}^{K(l \neq i)} \frac{1}{2|x_{cm}^i - x_{cm}^l|} \Big|_{x_{cm}^i = x_{cm}^i(t), x_{cm}^l = x_{cm}^l(t)}. \quad (109)$$

Now, consider the empirical distribution for the  $K$  particles  $f_K(x_{cm}, p_{cm}, t) := K^{-1} \sum_{i=1}^K \delta(x_{cm} - x_{cm}^i(t)) \delta(p_{cm} - p_{cm}^i(t))$  satisfying (in the sense of distributions) the Vlasov equation

$$\begin{aligned} \partial_t f_K + p_{cm} \frac{\partial}{\partial x_{cm}} f_K + \frac{\partial}{\partial p_{cm}} [F_K(x_{cm}, t) f_K] \\ = \frac{1}{K^2} \sum_{i=1}^K \frac{\partial}{\partial p_{cm}} \left[ \left( \frac{\partial}{\partial x_{cm}^i} \sum_{l=1}^{K(l \neq i)} \frac{M^2}{2|x_{cm}^i - x_{cm}^l|} \Big|_{x_{cm}^i = x_{cm}^i(t), x_{cm}^l = x_{cm}^l(t)} \right) \delta(x_{cm} - x_{cm}^i(t)) \delta(p_{cm} - p_{cm}^i(t)) \right], \end{aligned} \quad (110)$$

where

$$F_K(x_{cm}, t) := -\frac{\partial}{\partial x_{cm}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{M^2}{|x_{cm} - x'_{cm}|} f_K dx'_{cm} dp_{cm}. \quad (111)$$

We recall that Golse [21] and Bardos et al. [22, 23] considered a D-dimensional generalization of (106-111)<sup>5</sup>, for an arbitrary, symmetric, smooth interaction potential  $V$ , and showed that for  $K \rightarrow \infty$  one obtains the D-dimensional Vlasov-Poisson mean-field equations (see also section 4 of Part I). Thus the system (106-111), in the limit  $K \rightarrow \infty$ , is equivalent to the 2-dimensional Vlasov-Poisson mean-field equations (hereafter, writing  $x_{cm} = x$  and  $p_{cm} = p$ ):

$$\partial_t f(x, p, t) + \{H^{m.f.}(x, p, t), f(x, p, t)\} = 0, \quad (112)$$

$$H_{cl}^{m.f.}(x, p, t) := \frac{p^2}{2M} + \int_{\mathbb{R}} M \Phi_g^{m.f.}(x, x', t) dx'. \quad (113)$$

$$\frac{\partial^2 \Phi_g^{m.f.}}{\partial x^2} = 4\pi M \int_{\mathbb{R}} f(x, p, t) dp = 4\pi M \rho(x, t), \quad (114)$$

$$F(x, t) := -\frac{\partial}{\partial x} \int_{\mathbb{R}} M \Phi_g^{m.f.}(x, x', t) dx'. \quad (115)$$

Extending the above results to the D-dimensional case is formally straightforward.

## 6 Incorporating environmental decoherence

As discussed in subsection 2.1, environmental decoherence accompanied by effective collapse ensures that wavefunctions associated to macroscopic objects (composed of dBB/Nelsonian/ZSM particles) in the real world will not correspond to macroscopic quantum superpositions (i.e., involve strong quantum correlations). Thus it is reasonable to expect that wavefunctions associated to macroscopic objects in the real world will in general be WFPs. Though this expectation seems reasonable on general grounds, it would be even more convincing if we could demonstrate it in an explicit model of environmental decoherence in ZSM (or dBB). Here we sketch a suggestion for an explicit model.

<sup>5</sup>Regarding (106), Golse [21] and Bardos et al. [22, 23] take the empirical position distributions for the particles to be Dirac delta functions. The 1-dimensional Dirac delta function in position space is indeed a solution of (106), and note that the factors of (97) approach 1-dimensional Dirac delta functions as  $K \rightarrow \infty$ .

There exists a well-known model of generalized Brownian motion in classical nonequilibrium statistical mechanics called the Kac-Zwanzig (KZ) model [24, 25, 26] (the quantum mechanical analogue is the well-known Caldeira-Leggett model [27, 28]). The KZ model describes a heavy particle coupled to an external field and a heat bath, the bath modeled as an  $n$ -particle system of light harmonic oscillators, where the system particle couples bilinearly to each bath particle, with possibly frequency-dependent coupling strength. The classical Newtonian equations of motion for the system particle and bath particles are thereby coupled, and if one integrates out the bath variables, one finds, under the assumptions that the bath is at thermal equilibrium at temperature  $T$  and has arbitrary spectral density, a non-Markovian Langevin equation describing the time-evolution of the system particle.

Relatedly, Chou et al. [29] have shown that if one replaces the heavy probe particle of the KZ model with a system of  $N$  interacting identical harmonic oscillators, and if one assumes bilinear coupling of identical strength between the system and bath position coordinates, then there exists a canonical transformation that makes it possible to separate out the CM of the system from its relative degrees of freedom in the system-bath Hamiltonian. In other words, the transformed Hamiltonian, in the system degrees of freedom, is of the same form as the Schrödinger Hamiltonian in (36), the latter obtained from Oriols et al.'s coordinate transformation (12-13). Moreover, the transformed Hamiltonian entails that only the CM couples to the bath particles. Under the assumptions that (i) the system and bath are initially uncorrelated, (ii) the heat bath is initially at thermal equilibrium at temperature  $T$ , and (iii) the spectral density of the bath is arbitrary, Chou et al. then use the transformed Hamiltonian to define the unitary evolution of a system-bath density matrix. Tracing over the bath degrees of freedom, they find that the reduced density matrix for the system evolves by a non-Markovian master equation of Hu-Paz-Zhang type [30]. Such a master equation is, of course, well-known in the theory of quantum Brownian motion for open systems [31].

Our proposal, then, is to construct a KZ-type model from ZSM-Newton/Coulomb (or dBB-Newton/Coulomb), using the same starting assumptions as Chou et al., and applying the Oriols et al. scheme to the system and bath, respectively. This should make it possible to show that decoherence of the system wavefunction via interaction with the bath leads, under unitary evolution, to a macroscopic superposition of effectively orthogonal system-bath product states, and that such an evolution is accompanied by effective collapse of the system-bath configuration into one of the system-bath product states. In addition, the evolution of the system's CM particle position, with the bath variables integrated out, should be described by a non-Markovian modified Langevin equation, where the modifying terms are the quantum forces from the CM's quantum kinetic and the quantum kinetics of the relative degrees of freedom, and where both types of quantum kinetics are constructed from the effective system wavefunction to which the system configuration has collapsed. Then, taking the large particle number limits simultaneously for system and bath, it should be possible to show, by applying the arguments in section 3 of the present paper, that the equations of motion for the system and bath CM positions become effectively classical. In other words, we should recover the classical non-Markovian Langevin equation for the heavy particle in the classical KZ model.

The details of this proposal will be worked out in a stand-alone paper.

## 7 Conclusion

We have applied Oriols et al.'s large- $N$ -CM approximation scheme to a system of  $N$  identical, non-relativistic,  $z$  $bw$  particles interacting via potentials  $\hat{U}_{int}(\hat{x}_j - \hat{x}_k)$  and with external potentials  $\hat{U}_{ext}(\hat{x}_j)$ . This made it possible to: (i) self-consistently describe large numbers of identical  $z$  $bw$  particles interacting classical-gravitationally/electrostatically, without an independent particle approximation; (ii) avoid macroscopic semi-classical gravitational/electrostatic cat states and recover  $K$ -particle classical Newtonian gravity/electrodynamics for the CM descriptions of gravitationally/electrostatically interacting macroscopic particles (where the macroscopic particles are built out of interacting  $z$  $bw$  particles); and (iii) recover classical Vlasov-Poisson mean-field theory for macroscopic particles that interact gravitationally/electrostatically, in the weak-coupling large  $K$  limit. In addition, we have sketched a proposal for an explicit model of environmental decoherence consistent with the Oriols et al. large- $N$ -CM approximation scheme, the purpose of which is to explicitly demonstrate our claim that environmental decoherence plus effective collapse entails WFPs associated to real-world macroscopic objects.

We leave for future work the task of extending the ZSM-based large- $N$ -CM approximation scheme to relativistic massive particles and fields, in flat and curved spacetimes.

## References

- [1] M. Derakhshani. Semiclassical newtonian field theories based on stochastic mechanics i. 2017, <https://arxiv.org/abs/1701.06893>.
- [2] X. Oriols, D. Tena, and A. Benseny. Natural classical limit for the center of mass of many-particle quantum systems. 2016, <https://arxiv.org/abs/1602.03988>.
- [3] S. Goldstein. Stochastic mechanics and quantum theory. *Journal of Statistical Physics*, Vol. 47, Nos. 5/6., 47:645–667, 1987.
- [4] M. Jibu, T. Misawa, and K. Yasue. Measurement and reduction of wavefunction in stochastic mechanics. *Phys. Lett. A*, 150:59–62, 1990.
- [5] P. Blanchard, M. Cini, and M. Serva. *Ideas and Methods in Quantum and Statistical Physics*, chapter The measurement problem in the stochastic formulation of quantum mechanics. Cambridge University Press, Cambridge, 1992.
- [6] G. Peruzzi and A. Rimini. Quantum measurement in a family of hidden-variable theories. *Foundations of Physics Letters*, 9:505–519, 1996, <http://arxiv.org/abs/quant-ph/9607004>.
- [7] D. Dürr and S. Teufel. *Bohmian Mechanics: The Physics and Mathematics of Quantum Theory*. Springer, 2009.
- [8] S. Goldstein. Bohmian mechanics. *Stanford Encyclopedia of Philosophy*, 2013, <http://plato.stanford.edu/entries/qm-bohm/>.
- [9] V. Allori. *Decoherence and the Classical Limit of Quantum Mechanics*. PhD thesis, Università degli Studi di Genova, 2001, <http://www.niu.edu/~vallori/tesi.pdf>.
- [10] V. Allori, D. Dürr, S. Goldstein, and N. Zanghi. Seven steps towards the classical world. *Journal of Optics B: Quantum and semiclassical Optics, Volume 4, number 4*, 2002, <https://arxiv.org/abs/quant-ph/0112005>.
- [11] M. Derakhshani. A suggested answer to wallstrom’s criticism: Zitterbewegung stochastic mechanics ii. 2016, <https://arxiv.org/abs/1607.08838>.
- [12] R. Schiller. Quasi-classical theory of the nonspinning electron. *Phys. Rev.*, 125:1100, 1962.
- [13] N. Rosen. The relation between classical and quantum mechanics. *American Journal of Physics*, 32:597–600, 1964.
- [14] P. R. Holland. *The Quantum Theory of Motion*. Cambridge University Press, Cambridge, 1993.
- [15] P. Ghose. A continuous transition between quantum and classical mechanics. i. *Foundations of Physics vol. 32, pp. 871-892*, 32:871–892, 2002, <http://arxiv.org/abs/quant-ph/0104104>.
- [16] H. Nikolic. Classical mechanics without determinism. *Found. Phys. Lett.*, 19:553–566, 2006, <http://arxiv.org/abs/quant-ph/0505143>.
- [17] H. Nikolic. Classical mechanics as nonlinear quantum mechanics. *AIP Conf. Proc.*, 962:162–167, 2007, <http://arxiv.org/abs/0707.2319>.
- [18] M. Derakhshani. A suggested answer to wallstrom’s criticism: Zitterbewegung stochastic mechanics i. 2016, <http://arxiv.org/abs/1510.06391>.
- [19] D. Bohm and B. J. Hiley. *The Undivided Universe: An Ontological Interpretation of Quantum Theory*. Routledge, 1995.
- [20] A. Benseny, D. Tena, and X. Oriols. On the classical schroedinger equation. *Fluctuation and Noise Letters* 15, 1640011, 2016, <https://arxiv.org/abs/1607.00168>.

- [21] F. Golse. The mean-field limit for the dynamics of large particle systems. *Journées Equations aux dérivées partielles*, pp. 1-47, 2003, <https://eudml.org/doc/93451>.
- [22] C. Bardos, F. Golse, and N. Mauser. Weak coupling limit of the n-particle schroedinger equation. *Methods and Applications of Analysis*, Vol. 7, No. 2, pp. 275-294, 2000.
- [23] C. Bardos, L. Erdos, F. Golse, N. Mauser, and H. T. Yau. Derivation of the schroedinger-poisson equation from the quantum n-body problem. *C. R. Acad. Sci. Paris. Ser. I* 334, pp. 515-520, 2002.
- [24] G. W. Ford, M. Kac, and P. Mazur. Statistical mechanics of assemblies of coupled oscillators. *J. Math. Phys.* 6, 504, 1965.
- [25] R. Zwanzig. Nonlinear generalized langevin equations. *Journal of Statistical Physics*, Volume 9, Issue 3, pp. 215-220, 1973.
- [26] W. Eberling and I. M. Sokolov. *Statistical Thermodynamics and Stochastic Theory of Nonequilibrium Systems: Series on Advances in Statistical Mechanics - Volume 8*, chapter Chapter 7, Brownian Motion, subsection 7.7: A non-Markovian Langevin equation, page 155. World Scientific, 2005.
- [27] A. O. Caldeira and A. J. Leggett. Path integral approach to quantum brownian motion. *Physica* 121A: 587, 1983a.
- [28] A. O. Caldeira and A. J. Leggett. Quantum tunnelling in a dissipative system. *Annals of Physics* 149: 374, 1983b.
- [29] C-H. Chou, B. L. Hu, and T. Yu. Quantum brownian motion of a macroscopic object in a general environment. *Physica A* 387, 432, 2008, <https://arxiv.org/abs/0708.0882v1>.
- [30] B. L. Hu, J. P. Paz, and Y. Zhang. Quantum brownian motion in a general environment: Exact master equation with nonlocal dissipation and colored noise. *Phys. Rev. D* 45, 2843, 1992.
- [31] M. A. Schlosshauer. *Decoherence and the Quantum-to-Classical Transition (The Frontiers Collection)*. Springer, 2008.